

ERGODICITY OF \mathbb{Z}^2 EXTENSIONS OF IRRATIONAL ROTATIONS

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ABSTRACT. Let $\mathbf{T} = [0, 1)$ be the additive group of real numbers modulo 1, $\alpha \in \mathbf{T}$ be an irrational number and $t \in \mathbf{T}$. We consider skew product extensions of irrational rotations by \mathbb{Z}^2 determined by $T: \mathbf{T} \times \mathbb{Z}^2 \rightarrow \mathbf{T} \times \mathbb{Z}^2$
 $T(x, s_1, s_2) = \left(x + \alpha, \quad s_1 + 2\chi_{[0, \frac{1}{2})}(x) - 1, \quad s_2 + 2\chi_{[0, \frac{1}{2})}(x+t) - 1 \right)$. We study ergodic components of such extensions and use the results to display irregularities in the uniform distribution of the sequence $\mathbb{Z}\alpha$.

1. INTRODUCTION

The study of irrational rotations of the circle leads to various questions in number theory and ergodic theory. Let $\mathbf{T} = [0, 1)$ be the additive group of real numbers modulo 1. Fix an irrational $\alpha \in \mathbf{T}$ and let $t \in \mathbf{T}$ satisfy the condition that neither t nor $t + \frac{1}{2}$ is a multiple of $\alpha \pmod{1}$. Define a map $f: \mathbf{T} \rightarrow \mathbb{Z}$ by

$$(1.1) \quad f(x) = \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2}; \\ -1 & \text{for } \frac{1}{2} \leq x < 1 \end{cases}$$

and an irrational rotation T_0 of \mathbf{T} by

$$(1.2) \quad T_0 x = x + \alpha \pmod{1}.$$

Set $\mathbf{X} = \mathbf{T} \times \mathbb{Z}^2$ and define $T: \mathbf{X} \rightarrow \mathbf{X}$ by

$$(1.3) \quad T(x, s_1, s_2) = (x + \alpha, \quad s_1 + f(x), \quad s_2 + f(x+t)).$$

T is a skew product extension of irrational rotations on the circle by \mathbb{Z}^2 determined by $f(x)$ and t . We study ergodicity of T on \mathbf{X} relative to Haar measure, continuing a theme started by [5], [6] of Schmidt and by [7] of Veech. It is known that such property of skew product extensions of irrational rotations arises from irregularity of distribution of $\mathbb{Z}\alpha$. As for the case of cylinder flows, Oren in [4] gave complete solution to the problem of ergodicity of the map $F: \mathbf{T} \times E \rightarrow \mathbf{T} \times E$ defined by $F(x, s) = (x + \alpha, s + \mathbf{1}_{[0, \beta)}(x) - \beta)$, where $\beta \in \mathbf{T}$ and E is the closed subgroup of \mathbb{R} generated by 1 and β . Earlier, special cases were done by Schmidt for $\beta = \frac{1}{2}$, $\alpha = \frac{\sqrt{5}-1}{4}$ in [6] and for $\beta = \frac{1}{2}$, α irrational in [5]. Although ergodicity of cylinder flows has been understood thoroughly, due to the fact that $f(x)$ and $f(x+t)$ take on independent values, the situation of \mathbb{Z}^2 extensions of irrational rotations appear to be more complicated.

Note that by definition (1.3), we have

$$(1.4) \quad T^n(x, s_1, s_2) = (x + n\alpha, \quad s_1 + a_n(x), \quad s_2 + a_n(x+t)), \quad \forall n \in \mathbb{Z},$$

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where

$$(1.5) \quad a_n(x) = \begin{cases} \sum_{i=0}^{n-1} f(x + i\alpha) = 2 \sum_{i=0}^{n-1} \chi_{[0, \frac{1}{2})}(x + i\alpha) - n, & \forall n \geq 1; \\ 0, & \text{for } n = 0; \\ -a_{-n}(T_0^{-n}x), & \forall n \leq -1. \end{cases}$$

$t \in \mathbb{Z}\alpha$ and $t \in \mathbb{Z}\alpha + \frac{1}{2}$ are excluded *a priori*. To see this, note that for nonnegative integer m , $|a_n(x + m\alpha) - a_n(x)|$ is bounded by $2m$ because

$$(1.6) \quad |a_n(x + m\alpha) - a_n(x)| = \left| \sum_{i=0}^{m-1} f(x + n\alpha + i\alpha) - \sum_{i=0}^{m-1} f(x + i\alpha) \right|$$

$$(1.7) \quad \leq \sum_{i=0}^{m-1} |f(x + n\alpha + i\alpha)| + \sum_{i=0}^{m-1} |f(x + i\alpha)| \leq 2m, \quad \forall n > m.$$

We also have from (1.1) $f(x + \frac{1}{2}) = -f(x)$ and therefore

$$(1.8) \quad a_n(x + \frac{1}{2}) = -a_n(x), \quad \forall x \in \mathbf{T}, \quad \forall n.$$

$|a_n(x + \frac{1}{2} + m\alpha) + a_n(x)|$ is bounded from above by $2m$ thereof.

Also note that $a_n(x + t) \equiv a_n(x) \pmod{2}$. The parity $a_n(x)$ is always the same as that of n from (1.5). Hence T cannot be ergodic on the entire space \mathbf{X} . We set $G = \{(s_1, s_2) \in \mathbb{Z}^2 \mid s_1 \equiv s_2 \pmod{2}\}$. G is cocompact in \mathbb{Z}^2 .

$a_n(x)$ satisfies the additive cocycle equation

$$(1.9) \quad a_n(T_0^m x) - a_{n+m}(x) + a_m(x) = 0, \quad \forall m, n \in \mathbb{Z}, \quad \forall x \in \mathbf{T}.$$

Following [5, Definition 2.1] we have

Definition 1.1. $(a, t) : \mathbb{Z} \times \mathbf{T} \rightarrow \mathbb{Z}^2$ defined by

$$(1.10) \quad (a, t)(n, x) = (a_n(x), \quad a_n(x + t))$$

is called a cocycle for T_0 .

[5] showed that ergodicity of T , or equivalently, ergodicity of the cocycle (a, t) is determined by the group $\mathbb{E}^2(a, t)$ of essential values of (a, t) . Put $\overline{\mathbb{Z}^2} = \mathbb{Z}^2 \cup \{\infty\}$, the one point compactification of \mathbb{Z}^2 . We have the following definitions of essential values etc.

Definition 1.2. Let μ be Lebesgue measure on \mathbf{T} . An element $(k_1, k_2) \in \overline{\mathbb{Z}^2}$ is called an essential value of (a, t) if for every measurable set $A \subset \mathbf{T}$ with $\mu(A) > 0$, we have

$$(1.11) \quad \mu \left(\bigcup_{n \in \mathbb{Z}} \left(A \cap T_0^{-n} A \cap \{x \mid a_n(x) = k_1\} \cap \{x \mid a_n(x + t) = k_2\} \right) \right) > 0,$$

We denote the set of essential values of (a, t) by $\overline{\mathbb{E}^2}(a, t)$.

Definition 1.3. Set $\mathbb{E}^2(a, t) = \overline{\mathbb{E}^2}(a, t) \cap \mathbb{Z}^2$. $(k_1, k_2) \in \overline{\mathbb{E}^2}(a, t) \setminus \mathbb{E}^2(a, t)$ only if (k_1, k_2) does not lie in any compact subset of \mathbb{Z}^2 .

From [5] we derive the following properties

- (1) $\mathbb{E}^2(a, t)$ is a closed subgroup of \mathbb{Z}^2 under addition. $(k_1, k_2) \in \mathbb{E}^2(a, t)$ only if $k_1 \equiv k_2 \pmod{2}$.
- (2) (a, t) is a coboundary (that is, $a_n(x) = c(T_0^n x) - c(x)$ for a measurable map $c: \mathbf{T} \rightarrow \mathbb{Z}$) iff $\overline{\mathbb{E}^2}(a, t) = \{(0, 0)\}$.

We say that two cocycles $(a, t), (b, t): \mathbb{Z} \times \mathbf{T} \rightarrow \mathbb{Z}^2$ are cohomologous if $(a, t) - (b, t)$ is a coboundary. In this case $\overline{\mathbb{E}^2}(a, t) = \overline{\mathbb{E}^2}(b, t)$. Given a cocycle $(a, t): \mathbb{Z} \times \mathbf{T} \rightarrow \mathbb{Z}^2$, let $(a, t)^*: \mathbb{Z} \times \mathbf{T} \rightarrow \mathbb{Z}^2 / \mathbb{E}^2(a, t)$ be the corresponding quotient cocycle. We have the following important result from [5, Lemma 3.10]:

Lemma 1.4. $\mathbb{E}^2(a, t)^* = \{(0, 0)\}$.

We say that the cocycle (a, t) is regular if $\overline{\mathbb{E}^2}(a, t)^* = \{(0, 0)\}$. (a, t) is called nonregular if $\overline{\mathbb{E}^2}(a, t)^* = \{(0, 0), \infty\}$. If (a, t) is regular, then (a, t) is cohomologous to a cocycle $(b, t): \mathbb{Z} \times \mathbf{T} \rightarrow \mathbb{E}^2(a, t)$ and the latter is ergodic as a cocycle with values in the closed subgroup $\mathbb{E}^2(a, t)$ (see [5]). In particular, if $\mathbb{E}^2(a, t)$ is cocompact in \mathbb{Z}^2 then (a, t) is regular.

We utilize approach devised in [5], [4] to prove the following theorems:

Theorem 1.5. *For arbitrary irrational $\alpha \in \mathbf{T}$, the group of essential values $\mathbb{E}^2(a, t)$ of the cocycle (a, t) defined in (1.10) is $G = \{(s_1, s_2) \in \mathbb{Z}^2 \mid s_1 \equiv s_2 \pmod{2}\}$ for almost all $t \in \mathbf{T}$. In particular, (a, t) is regular for almost all $t \in \mathbf{T}$.*

Theorem 1.6. *If α is badly approximable, then the group of essential values $\mathbb{E}^2(a, t)$ is G if and only if $t \notin \mathbb{Z}\alpha$ and $t \notin \mathbb{Z}\alpha + \frac{1}{2}$.*

2. PERIOD APPROXIMATING SEQUENCES, PARTIAL CONVERGENTS AND OTHER PRELIMINARIES

For $x \in \mathbb{R}$ we denote the closest integer to x by $[x]$, denote $x - [x]$ by $\langle x \rangle$ and denote $|x - [x]|$ by $\|x\|$. We assume n to be nonnegative.

According to (1.5) $a_n(x)$ is locally constant except for points of discontinuities of $+2$ at $0, -\alpha, -2\alpha, \dots, -(n-1)\alpha$ and points of discontinuities of -2 at $\frac{1}{2}, \frac{1}{2} - \alpha, \frac{1}{2} - 2\alpha, \dots, \frac{1}{2} - (n-1)\alpha$. $a_n(x+t)$ is locally constant except for points of discontinuities of $+2$ at $-t, -t - \alpha, -t - 2\alpha, \dots, -t - (n-1)\alpha$ and points of discontinuities of -2 at $\frac{1}{2} - t, \frac{1}{2} - t - \alpha, \dots, \frac{1}{2} - t - (n-1)\alpha$.

If we set

$$(2.1) \quad S_n(x) = \sum_{i=0}^{n-1} \chi_{[0, \frac{1}{2})}(x + i\alpha) = \#\left\{i \mid 0 \leq i \leq n-1; \quad x + i\alpha \in [0, \frac{1}{2})\right\},$$

then from (1.5)

$$(2.2) \quad a_n(x) = 2S_n(x) - n.$$

The concept of essential values corresponds to that of periods in [4]. We have the following definition:

Definition 2.1. *A period approximating sequence is a sequence $\{(n_l, A_l)\}_{l=1}^\infty$ where*

- (1) $A_l \subset \mathbf{T}$, each A_l is measurable;
- (2) a_{n_l} is constant on both A_l and $A_l + t$, that is, $a_{n_l}(A_l) = k_1, a_{n_l}(A_l + t) = k_2 \quad \forall n_l$;
- (3) $\inf_l \mu(A_l) > 0$;
- (4) $\|n_l \alpha\| \rightarrow 0$.

The next lemma shows that a period approximating sequence defines an element in $\mathbb{E}^2(a, t)$.

Lemma 2.2. *If there exists a period approximating sequence $\{(n_l, A_l)\}_{l=1}^\infty$ such that $a_{n_l}(A_l) = k_1$, $a_{n_l}(A_l + t) = k_2$, $\forall n_l$, then $(k_1, k_2) \in \mathbb{E}^2(a, t)$.*

Proof. Set

$$B = \limsup_{l \rightarrow \infty} A_l = \bigcap_{l=1}^\infty \bigcup_{i=l}^\infty A_i.$$

$\mu(B) > 0$ because $\inf_l \mu(A_l) > 0$ and $\mu(\mathbf{T}) = 1$.

For arbitrary $A \subset \mathbf{T}$ with $\mu(A) > 0$, there exists $m \in \mathbb{Z}$ and $A' \subset A$ such that $\mu(A') > 0$ and $T_0^m A' \subset B$ because the action T_0 is ergodic. Hence

$$(2.3) \quad \mu\left(B \cap T_0^m A'\right) = \mu\left(\bigcap_{l=1}^\infty \bigcup_{i=l}^\infty (A_i \cap T_0^m A')\right) = \mu(T_0^m A') > 0,$$

hence there exists a subsequence $\{n'_l\}$ of $\{n_l\}$ such that for each n'_l , there exists a measurable set $A'_{n'_l} \subset A'$ with $\mu(A'_{n'_l}) > 0$ and

$$(2.4) \quad a_{n'_l}(T_0^m x) = k_1, \quad a_{n'_l}(T_0^m x + t) = k_2, \quad \forall x \in A'_{n'_l}.$$

Note that

$$(2.5) \quad \left| a_{n'_l}(T_0^m x) - a_{n'_l}(x) \right| = \left| \sum_{i=0}^{n'_l-1} f(x + i\alpha + m\alpha) - \sum_{i=0}^{n'_l-1} f(x + i\alpha) \right| \\ = \left| \sum_{i=0}^{m-1} f(x + i\alpha + n'_l\alpha) - \sum_{i=0}^{m-1} f(x + i\alpha) \right|,$$

(2.6)

$$\left| a_{n'_l}(T_0^m x + t) - a_{n'_l}(x + t) \right| = \left| \sum_{i=0}^{m-1} f(x + i\alpha + n'_l\alpha + t) - \sum_{i=0}^{m-1} f(x + i\alpha + t) \right|,$$

$$(2.7) \quad \lim \|n'_l\alpha\| = 0,$$

as well as the fact that m is fixed and depends on A only, we deduce that there exists some n'_l and $A'' \subset A' \subset A$ with $\mu(A'') > 0$ such that

$$(2.8) \quad a_{n'_l}(T_0^m x) = a_{n'_l}(x) = k_1, \quad a_{n'_l}(T_0^m x + t) = a_{n'_l}(x + t) = k_2, \quad \forall x \in A''.$$

Hence we have

$$(2.9) \quad \mu\left(A \cap T_0^{-n'_l} A \cap \left\{x \mid a_{n'_l}(x) = k_1\right\} \cap \left\{x \mid a_{n'_l}(x + t) = k_2\right\}\right) > 0.$$

$$(k_1, k_2) \in \mathbb{E}^2(a, t). \quad \square$$

We record the statement of the Denjoy-Koksma inequality [4, Lemma 2] here, which plays a fundamental role in the proof.

Lemma 2.3 (Denjoy-Koksma). *If $p \in \mathbb{N}, q \in \mathbb{N}$ satisfy*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2} \quad \text{and} \quad (p, q) = 1,$$

then $|a_q(x)| < 4$, $\forall x \in \mathbf{T}$, where $a_q(x)$ is defined in (1.5).

It follows from the proof of the above lemma that every interval of the form $\left[\frac{i}{q}, \frac{i+1}{q}\right)$ contains exactly one of the points $j\alpha$ for $0 \leq i, j \leq q-1$. In other words, the points $j\alpha$ ($0 \leq j \leq q-1$) are uniformly distributed on the unit circle.

We rely on numerous facts concerning continued fractions stated in texts such as [1]. A considerable portion of our approach is borrowed from [4]. However, here we need to construct period approximating sequence $\{(n_l, A_l)\}_{l=1}^\infty$ such that $a_{n_l}(A_l)$ and $a_{n_l}(A_l + t)$ take on *independent* values whereas predecessors of this paper only deal with cylinder flows.

We denote by $[a_0; a_1, a_2, \dots]$ the continued fraction of α and call the a_i the partial quotients of α . Denote by $\frac{p_k}{q_k}$ the k th partial convergent of α where $k \geq 0$. It is known from [1] that

$$(2.10) \quad \frac{p_k}{q_k} = [a_0; a_1, a_2, \dots, a_k];$$

$$(2.11) \quad \|q_k \alpha\| < \frac{1}{q_{k+1}} < \frac{1}{q_k};$$

$$(2.12) \quad \min_{q_k \leq q < q_{k+1}} \|q\alpha\| = \|q_k \alpha\| > \frac{1}{q_k + q_{k+1}} > \frac{1}{2q_{k+1}}.$$

$$(2.13) \quad q_k p_{k-1} - p_k q_{k-1} = (-1)^k.$$

Set

$$(2.14) \quad D(\alpha) = \left\{ q_k \mid \frac{p_k}{q_k} \text{ is a partial convergent of } \alpha \right\};$$

$$(2.15) \quad q^+ = \min\{q' \in D(\alpha) \mid q' > q\}, \quad \forall q \in D(\alpha).$$

Adopting arguments on [5, Page 229-230] we are able to prove the following lemma which constitutes the first step in the entire proof:

Lemma 2.4.

$$(2.16) \quad \mathbb{E}^2(a, t) \cap \{(1, 3), (1, -3), (1, 1), (1, -1), (3, 1), (3, -1), (3, 3), (3, -3)\} \neq \emptyset.$$

Proof. From (2.13) we derive that there are infinitely many odd $q \in D(\alpha)$. For such $q \in D(\alpha)$, the Denjoy-Koksma inequality applies. In addition, from (1.5) we see that $a_q(x)$ can only be odd, that is, $a_q(x)$ can only be ± 3 or ± 1 .

Consequently there exists a period approximating sequence $\{(q_l, A_l)\}_{l=1}^\infty$ such that $q_l \in D(\alpha)$,

- (1) $A_l \subset \mathbf{T}$;
- (2) a_{q_l} is constant on both A_l and $A_l + t$, $a_{q_l}(A_l) = k_1, a_{q_l}(A_l + t) = k_2 \quad \forall n_l$;
- (3) $\inf_l \mu(A_l) > 0$;
- (4) $\|q_l \alpha\| \rightarrow 0$

and $(k_1, k_2) \in \{(1, 3), (1, -3), (1, 1), (1, -1), (3, 1), (3, -1), (3, 3), (3, -3)\} \cup \{-(1, 3), -(1, -3), -(1, 1), -(1, -1), -(3, 1), -(3, -1), -(3, 3), -(3, -3)\}$. The proof is complete by noting that $\mathbb{E}^2(a, t)$ is a group under addition. \square

A major difficulty to prove Theorem 1.5 is therefore to show that $\mathbb{E}^2(a, t)$ is not isomorphic to \mathbb{Z} . We aim to show that $\mathbb{E}^2(a, t)$ is G for almost all t . This is done by using period approximating sequences. We derive from properties of continued fractions the following lemma:

Lemma 2.5. *For any nonzero $q \in D(\alpha)$, we have*

$$(2.17) \quad \min \{ \left\| \frac{1}{2} - j\alpha \right\| \mid |j| < q \} \geq \frac{1}{24q}.$$

Proof. We always have

$$(2.18) \quad \left\| \frac{1}{2} - j\alpha \right\| \geq \frac{\|2(\frac{1}{2} - j\alpha)\|}{2} = \frac{\|2j\alpha\|}{2}.$$

We consider five cases separately under the assumption that $0 < |j| < q$.

Case 1: $q^+ \geq 3q$, then since $\|2j| - q| < q$ from $0 < |j| < q$, we have $\|(|2j| - q)\alpha\| > \frac{1}{2q}$ from (2.12) and

$$(2.19) \quad \|2j\alpha\| = \|(|2j| - q)\alpha + q\alpha\| \geq \|(|2j| - q)\alpha\| - \|q\alpha\| > \frac{1}{2q} - \frac{1}{q^+} > \frac{1}{2q} - \frac{1}{3q} = \frac{1}{6q}.$$

Here we also used the inequality $\|q\alpha\| < \frac{1}{q^+}$ from (2.11).

Case 2: If $q^+ < 3q$ and $q^{++} < 3q$, then since $|2j| < 2q \leq q^{++}$, we have from (2.12)

$$(2.20) \quad \|2j\alpha\| \geq \|q^+\alpha\| \geq \frac{1}{2q^{++}} > \frac{1}{6q}.$$

Case 3: If $q^+ < 3q$, $q^{++} \geq 3q$ and $|q^+ - |2j|| < q$, then we have $\|(|2j| - q^+)\alpha\| > \frac{1}{2q}$ from (2.12) and

$$(2.21) \quad \|2j\alpha\| = \|(|2j| - q^+)\alpha + q^+\alpha\| \geq \|(|2j| - q^+)\alpha\| - \|q^+\alpha\| > \frac{1}{2q} - \frac{1}{q^{++}} > \frac{1}{2q} - \frac{1}{3q} = \frac{1}{6q}.$$

Case 4: If $q^+ < 3q$, $q^{++} \geq 3q$, $|q^+ - |2j|| \geq q$ and $|2j| \leq q$, then from (2.12) we get

$$(2.22) \quad \|2j\alpha\| \geq \|q\alpha\| > \frac{1}{2q^+} \geq \frac{1}{6q}.$$

Case 5: If $q^+ < 3q$, $q^{++} \geq 3q$, $|q^+ - |2j|| \geq q$ and $|2j| > q$, then

$$(2.23) \quad q^+ - |4j| < 3q - 2q = q, \quad 2q - q^+ > 2q - 3q = -q;$$

$$(2.24) \quad |2j| \leq q^+ - q \rightarrow q^+ - |4j| \geq q^+ - 2(q^+ - q) = 2q - q^+ > -q;$$

hence $|q^+ - |4j|| < q$ and from (2.12)

$$(2.25) \quad \|4j\alpha\| = \|(q^+ - |4j|)\alpha - q^+\alpha\| \geq \|(q^+ - |4j|)\alpha\| - \|q^+\alpha\| > \frac{1}{2q} - \frac{1}{q^{++}} \geq \frac{1}{6q};$$

and $\|2j\alpha\| \geq \frac{\|4j\alpha\|}{2}$. The inequality is established. \square

3. PROOF OF MAIN THEOREMS

Following [4] we set for each $q \in D(\alpha)$

$$(3.1) \quad \begin{aligned} \epsilon(q) &= q \cdot \min \{ \|-t - j\alpha\| \mid |j| < q \}; \\ \theta(q) &= q \cdot \min \{ \left\| \frac{1}{2} - t - j\alpha \right\| \mid |j| < q \}. \end{aligned}$$

We immediately derive that $\epsilon(q) < 1$ and $\theta(q) < 1$ from the proof of the Denjoy-Koksma inequality.

Proposition 3.1. *If*

$$(3.2) \quad \limsup_{\substack{q \in D(\alpha) \\ q \rightarrow \infty}} \min \{ \epsilon(q), \theta(q) \} > 0,$$

then $\mathbb{E}^2(a, t) = \{ (k_1, k_2) \in \mathbb{Z}^2 \mid k_1 \equiv k_2 \pmod{2} \} = G$.

Proof. Let $\{q_n\}_{n=1}^\infty \subset D(\alpha)$ be such that $\min \{ \epsilon(q_n), \theta(q_n) \} > \delta > 0, \forall n$.

Recall $a_{q_n}(x)$ as set in (1.5) is locally constant except for points of discontinuities of $+2$ at $0, -\alpha, -2\alpha, \dots, -(q_n - 1)\alpha$ and points of discontinuities of -2 at $\frac{1}{2}, \frac{1}{2} - \alpha, \frac{1}{2} - 2\alpha, \dots, \frac{1}{2} - (q_n - 1)\alpha$. $a_{q_n}(x + t)$ is locally constant except for points of discontinuities of $+2$ at $-t, -t - \alpha, -t - 2\alpha, \dots, -t - (q_n - 1)\alpha$ and points of discontinuities of -2 at $\frac{1}{2} - t, \frac{1}{2} - t - \alpha, \dots, \frac{1}{2} - t - (q_n - 1)\alpha$.

For fixed n , let $I_1, I_2, \dots, I_{4q_n}$ denote the intervals of constancy of both $a_{q_n}(x)$ and $a_{q_n}(x + t)$ in cyclic order. Since $a_{q_n}(\cdot)$ takes on at most four values by Lemma 2.3, there exists a union of intervals, A_n , such that $a_{q_n}(x)$ and $a_{q_n}(x + t)$ are constant on A_n and $\mu(A_n) \geq \frac{1}{16}$. Let A'_n be the union of intervals proximal on the right to those of A_n . Note that the distance between any discontinuities of $a_{q_n}(x)$ and $a_{q_n}(x + t)$ is given by $\|(i - j)\alpha\|$ or $\|\frac{1}{2} + (i - j)\alpha\|$ or $\| -t + (i - j)\alpha\|$ or $\|\frac{1}{2} - t + (i - j)\alpha\|$ for $0 \leq i, j \leq q_n - 1$. From (2.12), Lemma 2.5 and (3.2), we have that $\min \left\{ \frac{1}{24q_n}, \frac{\epsilon(q_n)}{q_n}, \frac{\theta(q_n)}{q_n} \right\}$ is a lower bound for the lengths $|I_i|$, $i = 1, 2, \dots, 4q_n$. Since every interval of length $\frac{2}{q_n}$ must contain a $+2$ discontinuity by discussion following Lemma 2.3, we have $|I_i| < \frac{2}{q_n}$. Therefore we have

$$(3.3) \quad \frac{|I_i|}{|I_j|} > \frac{1}{2} \min \left\{ \frac{1}{24}, \epsilon(q_n), \theta(q_n) \right\}, \quad 1 \leq i, j \leq 4q_n.$$

By setting $\epsilon = \min \left\{ \frac{1}{24}, \delta \right\}$, we thus have $\mu(A'_n) \geq \frac{1}{2}\epsilon\mu(A_n) \geq \frac{1}{32}\epsilon, \forall n$. $(a, t)(q_n, x) = (a_{q_n}(x), a_{q_n}(x + t))$ can take on A'_n only the values $(a_{q_n}(A_n) \pm 2, a_{q_n}(A_n + t))$ or $(a_{q_n}(A_n), a_{q_n}(A_n + t) \pm 2)$ since each interval of A'_n is proximal on the right to one of A_n . We can find $A''_n \subset A'_n$ such that $a_{q_n}(x)$ and $a_{q_n}(x + t)$ are both constant on A''_n , $\mu(A''_n) \geq \frac{1}{128}\epsilon$ and

$$(3.4) \quad (a_{q_n}(A''_n), a_{q_n}(A''_n + t)) = (a_{q_n}(A_n) \pm 2, a_{q_n}(A_n + t)) \text{ or } (a_{q_n}(A_n), a_{q_n}(A_n + t) \pm 2).$$

We assume that $a_{q_n}(A_n) = 1$ and $a_{q_n}(A_n + t) = 3$, that is $(1, 3)$ lies in $\mathbb{E}^2(a, t)$. We prove both $(2, 0)$ and $(0, 2)$ lie in $\mathbb{E}^2(a, t)$. Other possibilities can be treated analogously.

Case 1:

Suppose we have $(3, 3)$ and $(1, 3)$ both lie in $\mathbb{E}^2(a, t)$ as a result of the above arguments. $(\pm 2, 0)$ lies in $\mathbb{E}^2(a, t)$ because $\mathbb{E}^2(a, t)$ is a subgroup of \mathbb{Z}^2 .

Moreover, there exists a period approximating sequence $\{(q_n, A_n)\}_{n=1}^\infty$ which defines $(1, 3) \in \mathbb{E}^2(a, t)$. Namely we have

- (1) $A_n \subset \mathbf{T}$;
- (2) a_{q_n} is constant on both A_n and $A_n + t$, $a_{q_n}(A_n) = 1, a_{q_n}(A_n + t) = 3, \forall n$;
- (3) $\inf_n \mu(A_n) > 0$;
- (4) $\|q_n \alpha\| \rightarrow 0$.

Therefore there exists a period approximating sequence $\{(q'_n, B'_n)\}_{n=1}^\infty$ which defines $(k, 1) \in \mathbb{E}^2(a, t)$ for some $k \in \{\pm 1, \pm 3\}$. Namely we have

- (1) $\{q'_n\}$ is a subsequence of $\{q_n\}$, $B'_n + t \subset A'_n$, $\mu(B'_n) \geq \frac{1}{4}\mu(A'_n)$;

- (2) $a_{q'_n}$ is constant on both B'_n and $B'_n + t$, $a_{q'_n}(B'_n) = k$,
 $a_{q'_n}(B'_n + t) = a_{q'_n}(A'_n) = 1$, $\forall n'$;
- (3) $\inf_{n'} \mu(B'_n) > 0$;
- (4) $\|q'_n \alpha\| \rightarrow 0$.

$$(3.5) \quad (3, 3) \in \mathbb{E}^2(a, t) \text{ and } (2, 0) \in \mathbb{E}^2(a, t) \rightarrow (k, 3) \in \mathbb{E}^2(a, t);$$

$$(3.6) \quad (k, 1) \in \mathbb{E}^2(a, t) \text{ and } (k, 3) \in \mathbb{E}^2(a, t) \rightarrow (0, 2) \in \mathbb{E}^2(a, t).$$

Consequently both $(2, 0)$ and $(0, 2)$ lie in $\mathbb{E}^2(a, t)$.

Case 2:

Suppose we have $(-1, 3)$ and $(1, 3)$ both lie in $\mathbb{E}^2(a, t)$. $(\pm 2, 0)$ lies in $\mathbb{E}^2(a, t)$ because $\mathbb{E}^2(a, t)$ is a subgroup of \mathbb{Z}^2 .

Moreover, there exists a period approximating sequence $\{(q_n, A_n)\}_{n=1}^\infty$ which defines $(1, 3) \in \mathbb{E}^2(a, t)$. Namely we have

- (1) $A_n \subset \mathbf{T}$;
- (2) a_{q_n} is constant on both A_n and $A_n + t$, $a_{q_n}(A_n) = 1$, $a_{q_n}(A_n + t) = 3$, $\forall n$;
- (3) $\inf_n \mu(A_n) > 0$;
- (4) $\|q_n \alpha\| \rightarrow 0$.

Therefore there exists a period approximating sequence $\{(q'_n, B'_n)\}_{n=1}^\infty$ which defines $(k, 1) \in \mathbb{E}^2(a, t)$ for some $k \in \{\pm 1, \pm 3\}$. Namely we have

- (1) $\{q'_n\}$ is a subsequence of $\{q_n\}$, $B'_n + t \subset A'_n$, $\mu(B'_n) \geq \frac{1}{4}\mu(A'_n)$;
- (2) $a_{q'_n}$ is constant on both B'_n and $B'_n + t$, $a_{q'_n}(B'_n) = k$,
 $a_{q'_n}(B'_n + t) = a_{q'_n}(A'_n) = 1$, $\forall n'$;
- (3) $\inf_{n'} \mu(B'_n) > 0$;
- (4) $\|q'_n \alpha\| \rightarrow 0$.

$$(3.7) \quad (1, 3) \in \mathbb{E}^2(a, t) \text{ and } (2, 0) \in \mathbb{E}^2(a, t) \rightarrow (k, 3) \in \mathbb{E}^2(a, t);$$

$$(3.8) \quad (k, 1) \in \mathbb{E}^2(a, t) \text{ and } (k, 3) \in \mathbb{E}^2(a, t) \rightarrow (0, 2) \in \mathbb{E}^2(a, t).$$

Consequently both $(2, 0)$ and $(0, 2)$ lie in $\mathbb{E}^2(a, t)$.

Case 3:

Suppose we have $(1, 1)$ and $(1, 3)$ both lie in $\mathbb{E}^2(a, t)$. $(0, 2)$ lies in $\mathbb{E}^2(a, t)$. $(2, 2)$ also lies in $\mathbb{E}^2(a, t)$ and therefore $(2, 0)$ lies in $\mathbb{E}^2(a, t)$.

In all cases we have shown both $(2, 0)$ and $(0, 2)$ lie in $\mathbb{E}^2(a, t)$. Along with the assumption that $(1, 3)$ lies in $\mathbb{E}^2(a, t)$, we derive that $\mathbb{E}^2(a, t) = G$ as desired. \square

Remark 3.2. For arbitrary α the set of t satisfying (3.2) has full Lebesgue measure. Therefore for almost all $t \in \mathbf{T}$, we have $\mathbb{E}^2(a, t) = G$ and Theorem 1.5 is established.

Next we prove Theorem 1.6. Note that α is badly approximable if and only if its partial quotients are bounded.

Proposition 3.3. If α is badly approximable and

$$(3.9) \quad \lim_{\substack{q \in D(\alpha) \\ q \rightarrow \infty}} \min \{\epsilon(q), \theta(q)\} = 0,$$

then $t \in \mathbb{Z}\alpha$ or $t \in \mathbb{Z}\alpha + \frac{1}{2}$.

Proof. For each $q \in D(\alpha)$, let $|i_q| < q, |j_q| < q$ be such that

$$\epsilon(q) = q \|-t - i_q \alpha\|, \quad \theta(q) = q \|\tfrac{1}{2} - t - j_q \alpha\|.$$

Then we have from the assumption of the proposition

$$\lim_{\substack{q \in D(\alpha) \\ q \rightarrow \infty}} \min \{q \|-t - i_q \alpha\|, \quad q \|\tfrac{1}{2} - t - j_q \alpha\|\} = 0.$$

Because α is badly approximable, $\frac{q^+}{q}$ and $\frac{q^{++}}{q}$ have a uniform upper bound and

$$\lim_{\substack{q \in D(\alpha) \\ q \rightarrow \infty}} \min \{q^{++} \|-t - i_q \alpha\|, \quad q^{++} \|\tfrac{1}{2} - t - j_q \alpha\|\} = 0.$$

Also we have for arbitrary n_1 and n_2 the following inequalities:

$$(3.10) \quad \|n_1 \alpha - n_2 \alpha\| \leq \|-t - n_1 \alpha\| + \|-t - n_2 \alpha\|,$$

$$(3.11) \quad \|\tfrac{1}{2} + n_1 \alpha - n_2 \alpha\| \leq \|\tfrac{1}{2} - t - n_1 \alpha\| + \|-t - n_2 \alpha\|.$$

If we have $q^{++} \|-t - i_{q^+} \alpha\| < \frac{1}{100}$ and $q^{++} \|\tfrac{1}{2} - t - j_q \alpha\| < \frac{1}{100}$, then by (3.11)

$$q^{++} \|\tfrac{1}{2} + i_{q^+} \alpha - j_q \alpha\| < \frac{1}{50}.$$

Because

$$|i_{q^+} - j_q| \leq |i_{q^+}| + |j_q| < q^+ + q \leq q^{++},$$

this contradicts Lemma 2.5, which asserts that $q^{++} \|\tfrac{1}{2} + i_{q^+} \alpha - j_q \alpha\| \geq \frac{1}{24}$. Hence we have

$$\lim_{\substack{q \in D(\alpha) \\ q \rightarrow \infty}} q \|-t - i_q \alpha\| = 0 \quad \text{or} \quad \lim_{\substack{q \in D(\alpha) \\ q \rightarrow \infty}} q \|\tfrac{1}{2} - t - j_q \alpha\| = 0.$$

Suppose we have $\lim_{\substack{q \in D(\alpha) \\ q \rightarrow \infty}} q \|-t - i_q \alpha\| = 0$, then by (3.10)

$$\lim_{\substack{q \in D(\alpha) \\ q \rightarrow \infty}} q^{++} \|i_{q^+} \alpha - i_q \alpha\| = 0.$$

From (2.12) we derive that for q large enough $i_{q^+} = i_q$, that is, i_q is constant. Hence $t \in \mathbb{Z}\alpha$.

Suppose we have $\lim_{\substack{q \in D(\alpha) \\ q \rightarrow \infty}} q \|\tfrac{1}{2} - t - j_q \alpha\| = 0$, then

$$\lim_{\substack{q \in D(\alpha) \\ q \rightarrow \infty}} q^{++} \|j_{q^+} \alpha - j_q \alpha\| = 0.$$

From (2.12) we derive that for q large enough $j_{q^+} = j_q$, that is, j_q is constant. Hence $t \in \mathbb{Z}\alpha + \frac{1}{2}$. \square

Remark 3.4. When α is not badly approximable, Merrill [3] showed that if t belongs to an uncountable set of zero measure containing numbers well approximable by multiples of α , the cocycle $v = \chi_{[0,t)} - \chi_{[\frac{1}{2}, \frac{1}{2}+t)}$ is a coboundary. This implies $\mathbb{E}^2(a, t) = \{(k, k) \mid k \in \mathbb{Z}\}$. Similarly, If $t + \frac{1}{2}$ belongs to an uncountable set of zero measure containing numbers well approximable by multiples of α , then $\mathbb{E}^2(a, t) = \{(k, -k) \mid k \in \mathbb{Z}\}$.

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